First-order ordinary differential equations: separable variables

Introduction to Separable Variables

A first-order ODE (meaning its highest derivative is $y'(x)$) of the form

$$y' = f(x) \cdot g(y)$$

can be solved using the method of "separable variables." This method allows us to rearrange the terms so that all expressions involving y are on one side of the equation and all expressions involving x are on the other.

Since $y' \equiv \frac{dy}{dx}$, we have:

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

$$\Rightarrow \quad \frac{1}{g(y)}\, dy = f(x)\, dx$$

Now, with each variable on a different side of the equation, applying integrals we arrive at the solution:

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

$$\Rightarrow \quad \varphi(y) = \theta(x) + C$$

Note: From this point on, in exercises we will omit the variable (x) in expressions $y(x), y'(x), \ldots$, as we already know that $y = f(x)$.

Example

$$3x\, y' - x^2 y = 0$$

First, we rearrange the equation:

$$3x\, y' = x^2 y \quad \Rightarrow \quad y' = \frac{xy}{3}$$

Then, since $y' = \frac{dy}{dx}$, we rewrite as:

$$\frac{dy}{dx} = \frac{xy}{3}$$

Separating the variables and integrating, we obtain:

$$\frac{dy}{y} = \frac{x}{3}\, dx \quad \Rightarrow \quad \int \frac{1}{y}\, dy = \int \frac{x}{3}\, dx$$

Therefore, solving the integrals:

$$\ln y = \frac{x^2}{6} + C$$

Continuing from the previous example, we find the general solution.

$$\ln y = \frac{x^2}{6} + C \quad \Rightarrow \quad y = e^{\frac{x^2}{6}+C} \quad \Rightarrow \quad y = e^{\frac{x^2}{6}} \cdot e^C$$

If we let $e^C = k$, then the general solution of the given ODE is:

$$\boxed{y(x) = ke^{\frac{x^2}{6}}}$$

Obtaining a Particular Solution

If we want to obtain a particular solution, we need an initial condition (I.C.) to find the value of k. Suppose $y(6) = 1$:

$$y(x) = ke^{\frac{x^2}{6}} \quad \Rightarrow \quad y(6) = ke^{\frac{6^2}{6}} = 1 \quad \Rightarrow \quad k = \frac{1}{e^6}$$

Thus, with the given I.C., the particular solution is:

$$y^p(x) = \frac{1}{e^6}e^{\frac{x^2}{6}}$$